ANALYTIC PROPERTIES OF FEYNMAN DIAGRAMS IN QUANTUM FIELD THEORY

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In 1966, when the first (Russian) edition of this book was published, two other books on the same subject [50, 87] appeared.

The book by Eden *et al.* [50] represents a physicist survey of the analytic properties of the S-matrix. It contains the main physical ideas and results, but does not give much attention to mathematical proofs and detailed calculations.

The book by Hwa and Teplitz [87], on the other hand, introduces the reader to the more mathematically oriented work of recent years, which uses such modern methods as algebraic topology and homology theory. (It includes a number of reprints on the subject.) Work in this direction has multiplied in recent years (see in particular refs. [63, 108, 164–5, 167–8])

The present book is in an intermediate position between the two. Its methods are elementary and do not touch on the applications of homology theory, but in contrast to [50] it treats the problems in considerable detail.

Most of the book is devoted to a self-contained exposition of the majorization method in the study of the analytic properties of Feynman graphs; this is applied to the derivation of single variable dispersion relations. A special chapter (4) deals with the study of the singularity surface of an arbitrary diagram. A number of examples for which the Landau curves can be found explicitly are treated in detail. A new section is added to the English edition in this chapter; it illustrates the Cutkosky rules and the Mandelstam representation with the example of a fourth-order diagram. The derivation of fixed angle dispersion relations for the pion–nucleon scattering amplitude in Chapter 3 of the first edition was complicated and incomplete, so it has been eliminated in the present edition. Also, a number of minor corrections has been added throughout and the list of references has been extended.

Preface to the English Edition

I would like to thank Dr. Clifford Risk for taking both the initiative and the laborious job of the English translation. He contributed to the elimination of some errors of the original and added a number of clarifying remarks. I am very grateful to Professor Stapp for his critical remarks which helped improve the final version of the translation. The work on the English edition was done during my stay at the Institute for Advanced Study in Princeton, where I enjoyed the hospitality of Professor C.Kaysen and of the Faculty of Natural Sciences. It is also a pleasure to thank Professor T.Regge for his kind interest in the work.

Princeton, February 1970 I. TODOROV

Translator's Note

DURING the last four years several books have appeared summarizing various areas in which the study of analytic properties of the scattering amplitude $A(s, t)$ has been developed. The book by Eden *et al.* [50] (chapters 1, 2) develops the analytic properties in s and t as obtained in perturbation theory; this is motivated by the Mandelstam double dispersion conjecture. The book by Hwa and Teplitz [87] summarizes how the techniques used in [50] can be put on a rigorous foundation through homology theory. About a decade ago the analytic properties of $A(s, t)$ were studied from the general framework of quantum field theory, and single variable dispersion relations were proven (see Bogoliubov and Shirkov [14] and also the summary in *Dispersion Relations*, G. R. Screaton, Oliver & Boyd, London, 1960); recently, Martin [125] explored the unitarity condition in field theory, and in particular obtained a larger domain of analyticity of the scattering amplitude. The book by R. J. Eden, *High Energy Collisions of Elementary Particles*, Cambridge University Press, 1967, summarizes the phenomenological applications of all of these results.

However, there has remained a need for a book that summarizes the analytic properties obtained from perturbation theory by the majorization technique and the Symanzik theorem. The present work fulfills this need. Starting with an integral representation for a Feynman amplitude (Chapter 1), diagrams are found for individual processes which have the smallest domains of analyticity. The number of these (majorizing) diagrams is reduced with the Symanzik theorem (Chapter 2), their analytic properties are studied, and eventually one can derive dispersion relations for the amplitudes of a large number of processes (Chapter 3). In Chapter 4 the proper singularities of several diagrams are studied, and the Cutkosky rules formulated. The book contains some new material; the reader will be surprised to learn of the existence of a pole in the box diagram amplitude on the physical sheet.

Undertaking the translation was first suggested to me by Dr. David Olive, and further encouraged by Professor R. J. Eden and Professor Marc Ross. During the final stages of the translation, we were fortunate to have Professor Henry Stapp read through the manuscript and suggest many improvements and revisions, for which we are very grateful. We express our thanks to Pergamon Press for their patience and assistance during the long period of preparation of the text.

The translation was supported in part by funds provided by the A.E.C. I am grateful to Professor Marc Ross for the hospitality of the Physics Department at the University of Michigan, and to Professor G. Chew for the hospitality of the Lawrence Radiation Laboratory as an A.E.C. Fellow.

Berkeley, California
April 1970

Foreword

THE present book deals with a comparatively new branch of quantum field theory.

The systematic study of the analytic properties of the matrix elements of perturbation theory began with the work of Nambu and Symanzik in 1957–8. In the following years this problem occupied the attention of elementary particle physicists and became the subject of many investigations.

As has often happened in recent years, new problems and fads attracted theoreticians before the ultimate question of whether the Mandelstam representation is valid to any order of perturbation theory had been answered.

Lately there has been a relative quiet in the development of the area. It seems a proper time to summarize the basic results that were obtained during five years of intensive work.

This monograph is a revised and completed version of an earlier work by the author [189], which was published as a preprint by the Joint Institute for Nuclear Research in Dubna. Naturally, the selection of material has reflected the interests of the author and his participation in the study of the problems covered in the book. In any event, the book does not pretend to be an exhaustive survey of all work in the area.

The book is primarily directed toward readers familiar with the fundamentals of quantum field theory as presented, for example, in the first four chapters of the book by Bogoliubov and Shirkov [14]. However, all the basic concepts that we will be dealing with in the systematic presentation which begins in Chapter 1 are defined in the text. Therefore, the author hopes that the book will be also accessible to mathematicians interested in mathematical problems of modern physics.

The author's interest in the problems considered in this book arose during joint work with Professor A. A. Logunov and Dr. H. A. Chernikov. In some stages of the work occupying the central part of the book Liu Yi-Chen, M. A. Mestvirishvili and A. N. Tavkhelidze also participated. The results were repeatedly discussed with Professors N. N. Bogoliubov, V. S. Vladimirov, and O. S. Parasiuk. The author has benefited from helpful discussions with Professor M. G. Krein. To all of them I express my deep gratitude.

Sofia, October 1965 THE AUTHOR

Introduction

1. Dispersion relations and perturbation theory

The fundamental quantity in the study of particle interactions in quantum field theory is the scattering matrix S. The square moduli of its elements

$$S(p_1, \ldots, p_i; q_1, \ldots, q_j) = S(p_1, \ldots, p_i, -q_1, \ldots, -q_j)$$

give the probability of transition from the state of i free particles and momenta $p_1, \ldots, p_i$ at time $t \to -\infty$ to the state of j free particles with momenta $q_1, \ldots, q_j$ at time $t \to +\infty$.

In perturbation theory the S-matrix elements are expanded in a power series in the coupling constant g

$$S(p_1, \ldots, p_N) = \sum_{n=0}^{\infty} g^n S_n(p_1, \ldots, p_N). \tag{1}$$

In electrodynamics, $g^2 = 1/137$, and the expansion has proven to be of practical use. However, for strong interactions $g^2 \approx 15$, and, as we might have expected, the first few terms of (1) do not fit experiment at all.

In the last decade much progress has been made in studying strong interactions by using the dispersion relation techniques. The basic idea is to investigate the analytic properties of the S-matrix elements for complex values of momenta. Such an investigation was first carried out starting from the general principles of quantum field theory: micro-causality, relativistic invariance of the Lorentz group, and the existence of a complete system of physical states with positive energy (the spectral property). An essential feature of these postulates is that they define a linear space of functions; we will call them the "linear postulates". As a rule, unitarity, which is a non-linear property, is not used in deriving the analytic properties of S-matrix elements.* On the basis of the linear properties, Bogoliubov *et al.* [14, 16] derived dispersion relations in the energy variable for physical values of the momentum transfer in the physically important case of pion–nucleon scattering. Later on it became clear that Bogoliubov's method is quite general and can be applied in principle to all processes involving two incident and two outgoing particles [17, 146, 176, 187, 191]. Meanwhile, at one essential stage in the proof—the analytic continuation of the absorptive part of the amplitude—the methods of the theory of functions of several complex variables [22, 191, 193–4] and the Dyson representation of the causal commutator [47, 193–4] were used to strengthen the original result for π–N scattering. Lehmann [109] showed that the π–N scattering amplitude is analytic in a certain ellipse in the complex plane of momentum transfer $\Delta^2 = (-t/4)$. By an analogous method the analytic properties of the vacuum expectation value of the product of three local fields were analyzed [93] and an integral representation was found for this matrix element [94].†

However, shortly after all these successes, limitations of the general method were observed. It was shown [22] that using only causality and the spectral property and without involving the symmetry properties, dispersion relations could be derived for the N–N scattering amplitude only for masses of the pion (m) and nucleon (M) which satisfy

$$m \geqslant (\sqrt{2} - 1)\, M. \tag{2}$$

This relation is not valid for the experimental masses. At the moment it is not clear if the symmetry conditions allow further analytic continuation of the amplitude. However, Jost [91] constructed an example for the meson–nucleon vertex function in which he used all the linear properties,

* Cf., however, [4, 106, 107, 124], where the first steps in such a use of the unitarity condition were made. In [162] the consequences of the unitary condition are studied in combination with a certain "principle of maximal analyticity", which does not seem to have a clear mathematical formulation. Significant progress in this direction is achieved in the recent work of Martin [125].

† A systematic survey of work on the analytic properties of quantum amplitudes (based on the general linear postulates) is given in [17]. (There is also a bibliography in this reference.)

including the symmetry requirement. In this example it was clear that these postulates do not guarantee the validity of dispersion relations when

$$m < \left(\frac{2}{\sqrt{3}} - 1\right) M;$$

this relation is satisfied by the experimental values of the pion and nucleon masses.

The only method that has been developed up to now which systematically uses the non-linear property of unitarity of the S-matrix (along with other postulates) is the perturbation theory. This "theory" is incomplete in many ways: the coefficients S_n of the formal series (1) are sums of multiple integrals, most of which diverge. After "renormalization" of the divergent integrals, which leads to finite values of S_n, it is still doubtful if the series (1) itself converges. Finally, for strong interactions where $g > 1$, there is little hope in the practical use of an expansion in powers of g even if the series (1) asymptotically converges. All this led in the mid-1950's to the (precipitate) conclusion that the perturbation expansion is of no use whatsoever as far as strong interactions are concerned.

The difficulties that arose from the study of the analytic properties of matrix elements led physicists to look at the perturbation theory series from a new point of view. It was shown that the individual terms of the series have analytic properties consistent with the general principles, their singularities having a simple interpretation. Moreover, by studying the analytic properties of individual Feynman diagrams the origin of inequalities like eq. (2) that occur in the general approach was understood [96] (see section 1 of Chapter 3). On the basis of the analysis of the analytic properties of simple scattering diagrams, Mandelstam [122] advanced his famous hypothesis about the double spectral representation of the scattering amplitude. (It is a generalization of a representation postulated by Nambu [143], which had been shown to be violated in perturbation theory.) The question then arose of finding a domain G in which every term of the perturbation series for the scattering amplitude would be analytic. Of course, to prove analyticity of the whole amplitude in the domain G one ought to show that the series converges uniformly in this region.* However, in numerous articles that dealt with the study of analytic properties of the amplitudes in perturbation theory, the question of convergence of the power series in g was not touched on at all. In other papers that were specifically devoted to this question various unproven assertions were stated. In [64, 130] field theory models were considered which had series expansion in powers of charge with a finite radius of convergence.†

Confidence in the perturbation method was also partly restored by the work on the renormalization of individual terms in the series [15, 23, 26, 27, 69, 85, 86, 88, 149, 179–80, 198, 206]; in these articles a precise mathematical meaning was given to the procedure of removing the divergences.

Finally, we mention the study of the asymptotic behavior of Feynman graphs for high energy begun in 1962 [4, 51–3, 56, 75, 76, 163, 186, 205–6]. Recently, methods of summing diagrams in the high-energy limit (including all "crossed ladders") were developed and led to the so-called eikonal approximation, which appears to be of particular use in quantum electrodynamics (see, for example, [1, 6, 24, 28–33, 111]). These results, just like the results of the study of analytic properties in perturbation theory, are certainly not rigorous and complete (since the contribution from the remaining diagrams is not estimated). However, taken together all these arguments reinforce the confidence of physicists that the perturbation theory series does contain useful information about strong interactions also.

2. A survey of work on the analytic properties of S-matrix elements in perturbation theory

It is difficult to sort out the many articles that have been written on the analytic properties of Feynman diagrams. Not only is there a large number

* According to a well-known theorem of Vitali (see any textbook on the theory of analytic functions) to do this it is sufficient to show that there is a uniform bound of the partial sums of the series inside of the region G and that the series has an ordinary convergence in some subset $E \subset G$ which contains at least one limit point in G.

† In [130] there are also references to the previous works, which contain arguments for convergence (or divergence) of the perturbation theory series. More recently the problem was studied in [7, 178].

of articles on the subject (the number has especially grown since 1959), but, furthermore, the different publications have no common mathematical standard. Many articles that sometimes claim a very strong result in reality contain only vague, unproven assertions and frequently outright mistakes. Naturally, this survey of the principal directions of work in the area does not purport to give a critical analysis of all the literature on the subject.

We divide the works on the study of analytic properties of the matrix elements of perturbation theory into two groups.

In the first group we put articles that examine the simplest lower order diagrams of a given process. Strictly speaking, this type of investigation can produce only a negative result; namely, it can only show that a given hypothesis is *not* true in the lowest orders of perturbation theory. However, these studies of specific examples do elucidate a number of characteristic features in the analytic behavior of Feynman integrals and provide a basis for developing general methods of studying analytic properties of diagrams. As an example of the pioneer role they performed we mention the work of Karplus *et al.* [96, 97]. They investigated the simplest, non-trivial diagrams for the vertex part and the scattering amplitude. The important concept of an anomalous threshold* was introduced and a graphical method was given for finding the proper singularities of a given diagram by means of constructing the so-called dual diagram. The method of dual diagrams was developed and systematically used in [104, 148, 183]. A series of examples of locating the real singularities of more complicated diagrams was examined in [100–1, 118, 148, 152, 172]. In [104, 173] the question of the nature of these singularities was discussed. In the majority of these articles the simplifying assumption is made that all particles are scalar. (This does not affect the location of the singularities—compare, for example, [139] or Chapter 1, section 1.3 of this book.) An example of a diagram for nucleon–nucleon scattering in which the spinor structure of the nucleon propagator is taken into account is considered in detail in [77].

It is considerably more complicated to find the complex singularities of

* Nambu [145] and Oehme [147] arrived at this concept independently. Anomalous thresholds and their physical meaning were discussed later in [11, 12, 43]. For the definitions of normal and anomalous threshold see Chapter 3, section 1.4.