A COURSE IN MATHEMATICAL ANALYSI

FUNCTIONS OF A COMPLEX VARIABLE

BEING PART I OF VOLUME II

BY

ÉDOUARD GOURSAT

PROFESSOR OF MATHEMATICS, THE UNIVERSITY OF PARIS

TRANSLATED BY

EARLE RAYMOND HEDRICK

PROFESSOR OF MATHEMATICS, THE UNIVERSITY OF MISSOURI

AND

OTTO DUNKEL

ASSOCIATE PROFESSOR OF MATHEMATICS, WASHINGTON UNIVERSITY

GINN AND COMPANY

BOSTON · NEW YORK · CHICAGO · LONDON

ATLANTA · DALLAS · COLUMBUS · SAN FRANCISCO

PRINTED IN THE UNITED STATES OF AMERICA

236.7

The Athenæum Press
GINN AND COMPANY · PROPRIETORS · BOSTON · U.S.A.

AUTHOR'S PREFACE — SECOND FRENCH EDITION

The first part of this volume has undergone only slight changes, while the rather important modifications that have been made appear only in the last chapters.

In the first edition I was able to devote but a few pages to partial differential equations of the second order and to the calculus of variations. In order to present in a less summary manner such broad subjects, I have concluded to defer them to a third volume, which will contain also a sketch of the recent theory of integral equations. The suppression of the last chapter has enabled me to make some additions, of which the most important relate to linear differential equations and to partial differential equations of the first order.

E. GOURSAT

TRANSLATORS' PREFACE

As the title indicates, the present volume is a translation of the first half of the second volume of Goursat's "Cours d'Analyse." The decision to publish the translation in two parts is due to the evident adaptation of these two portions to the introductory courses in American colleges and universities in the theory of functions and in differential equations, respectively.

After the cordial reception given to the translation of Goursat's first volume, the continuation was assured. That it has been delayed so long was due, in the first instance, to our desire to await the appearance of the second edition of the second volume in French. The advantage in doing so will be obvious to those who have observed the radical changes made in the second (French) edition of the second volume. Volume I was not altered so radically, so that the present English translation of that volume may be used conveniently as a companion to this; but references are given here to both editions of the first volume, to avoid any possible difficulty in this connection.

Our thanks are due to Professor Goursat, who has kindly given us his permission to make this translation, and has approved of the plan of publication in two parts. He has also seen all proofs in English and has approved a few minor alterations made in translation as well as the translators' notes. The responsibility for the latter rests, however, with the translators.

E. R. HEDRICK
OTTO DUNKEL

CONTENTS

A COURSE IN MATHEMATICAL ANALYSIS

VOLUME II. PART I

THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

CHAPTER I

ELEMENTS OF THE THEORY

I. GENERAL PRINCIPLES. ANALYTIC FUNCTIONS

1. Definitions. An *imaginary quantity*, or *complex quantity*, is any expression of the form $a + bi$ where a and b are any two real numbers whatever and i is a special symbol which has been introduced in order to generalize algebra. Essentially a complex quantity is nothing but a system of two real numbers arranged in a certain order. Although such expressions as $a + bi$ have in themselves no concrete meaning whatever, we agree to apply to them the ordinary rules of algebra, with the additional convention that i^2 shall be replaced throughout by -1.

Two complex quantities $a + bi$ and $a' + b'i$ are said to be equal if $a = a'$ and $b = b'$. The sum of two complex quantities $a + bi$ and $c + di$ is a symbol of the same form $a + c + (b + d)i$; the difference $a + bi - (c + di)$ is equal to $a - c + (b - d)i$. To find the product of $a + bi$ and $c + di$ we carry out the multiplication according to the usual rules for algebraic multiplication, replacing i^2 by -1, obtaining thus

$$(a + bi)(c + di) = ac - bd + (ad + bc)i.$$

The quotient obtained by the division of $a + bi$ by $c + di$ is defined to be a third imaginary symbol $x + yi$, such that when it is multiplied by $c + di$, the product is $a + bi$. The equality

$$a + bi = (c + di)(x + yi)$$

is equivalent, according to the rules of multiplication, to the two relations

$$cx - dy = a, \qquad dx + cy = b,$$

whence we obtain

$$x = \frac{ac + bd}{c^2 + d^2}, \qquad y = \frac{bc - ad}{c^2 + d^2}.$$

The quotient obtained by the division of $a + bi$ by $c + di$ is represented by the usual notation for fractions in algebra, thus,

$$x + yi = \frac{a + bi}{c + di}.$$

A convenient way of calculating x and y is to multiply numerator and denominator of the fraction by $c - di$ and to develop the indicated products.

All the properties of the fundamental operations of algebra can be shown to apply to the operations carried out on these imaginary symbols. Thus, if A, B, C, $\cdots$ denote complex numbers, we shall have

$$A \cdot B = B \cdot A, \quad A \cdot B \cdot C = A \cdot (B \cdot C), \quad A(B + C) = AB + AC, \quad \cdots$$

and so on. The two complex quantities $a + bi$ and $a - bi$ are said to be *conjugate imaginaries*. The two complex quantities $a + bi$ and $-a - bi$, whose sum is zero, are said to be *negatives* of each other or *symmetric* to each other.

Given the usual system of rectangular axes in a plane, the complex quantity $a + bi$ is represented by the point M of the plane xOy, whose coördinates are $x = a$ and $y = b$. In this way a concrete representation is given to these purely symbolic expressions, and to every proposition established for complex quantities there is a corresponding theorem of plane geometry. But the greatest advantages resulting from this representation will appear later. Real numbers correspond to points on the x-axis, which for this reason is also called the *axis of reals*. Two conjugate imaginaries $a + bi$ and $a - bi$ correspond to two points symmetrically situated with respect to the x-axis. Two quantities $a + bi$ and $-a - bi$ are represented by a pair of points symmetric with respect to the origin O. The quantity $a + bi$, which corresponds to the point M with the coördinates (a, b), is sometimes called its *affix*.* When there is no danger of ambiguity, we shall denote by the same letter a complex quantity and the point which represents it.

Let us join the origin to the point M with coördinates (a, b) by a segment of a straight line. The distance OM is called the *absolute value* of $a + bi$, and the angle through which a ray must be turned from Ox to bring it in coincidence with OM (the angle being measured, as in trigonometry, from Ox toward Oy) is called the *angle* of $a + bi$.

* This term is not much used in English, but the French frequently use the corresponding word *affixe*. — TRANS.

Let ρ and ω denote, respectively, the absolute value and the angle of $a+bi$; between the real quantities a, b, ρ, ω there exist the two relations $a=\rho\cos\omega$, $b=\rho\sin\omega$, whence we have

$$\rho=\sqrt{a^2+b^2},\qquad \cos\omega=\frac{a}{\sqrt{a^2+b^2}},\qquad \sin\omega=\frac{b}{\sqrt{a^2+b^2}}.$$

The absolute value ρ, which is an essentially positive number, is determined without ambiguity; whereas the angle, being given only by means of its trigonometric functions, is determined except for an additive multiple of 2π, which was evident from the definition itself. Hence every complex quantity may have an infinite number of angles, forming an arithmetic progression in which the successive terms differ by 2π. In order that two complex quantities be equal, their absolute values must be equal, and moreover their angles must differ only by a multiple of 2π, and these conditions are sufficient. The absolute value of a complex quantity z is represented by the same symbol $|z|$ which is used for the absolute value of a real quantity.

Let $z=a+bi$, $z'=a'+b'i$ be two complex numbers and m, m' the corresponding points; the sum $z+z'$ is then represented by the point m'', the vertex of the parallelogram constructed upon Om, Om'. The three sides of the triangle $Om\,m''$ (Fig. 1) are equal respectively to the absolute values of the quantities z, z', $z+z'$. From this we conclude that *the absolute value of the sum of two quantities is less than or at most equal to the sum of the absolute values of the two quantities, and greater than or at least equal to their difference.* Since two quantities that are negatives of each other have the same absolute value, the theorem is also true for the absolute value of a difference. Finally, we see in the same way that the absolute value of the sum of any number of complex quantities is at most equal to the sum of their absolute values, the equality holding only when all the points representing the different quantities are on the same ray starting from the origin.

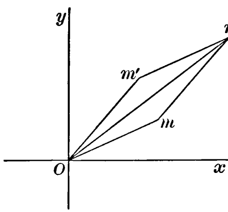

Fig. 1

If through the point m we draw the two straight lines mx' and my' parallel to Ox and to Oy, the coördinates of the point m' in this system of axes will be $a'-a$ and $b'-b$ (Fig. 2). The point m' then represents $z'-z$ in the new system; the absolute value of

$z' - z$ is equal to the length mm', and the angle of $z' - z$ is equal to the angle θ which the direction mm' makes with mx'. Draw through O a segment Om_1 equal and parallel to mm'; the extremity m_1 of this segment represents $z' - z$ in the system of axes Ox, Oy. But the figure $Om'm_1$ is a parallelogram; the point m_1 is therefore the symmetric point to m with respect to c, the middle point of Om'.

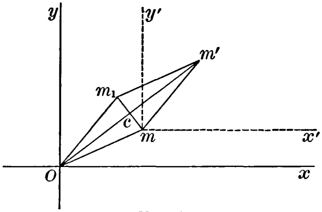

FIG. 2

Finally, let us obtain the formula which gives the absolute value and angle of the product of any number of factors. Let

$$z_k = \rho_k(\cos\omega_k + i\sin\omega_k), \qquad (k = 1, 2, \cdots, n),$$

be the factors; the rules for multiplication, together with the addition formulæ of trigonometry, give for the product

$$z_1 z_2 \cdots z_n = \rho_1\rho_2\cdots\rho_n[\cos(\omega_1 + \omega_2 + \cdots + \omega_n) + i\sin(\omega_1 + \omega_2 + \cdots + \omega_n)],$$

which shows that *the absolute value of a product is equal to the product of the absolute values, and the angle of a product is equal to the sum of the angles of the factors.* From this follows very easily the well-known formula of De Moivre:

$$\cos m\omega + i\sin m\omega = (\cos\omega + i\sin\omega)^m,$$

which contains in a very condensed form all the trigonometric formulæ for the multiplication of angles.

The introduction of imaginary symbols has given complete generality and symmetry to the theory of algebraic equations. It was in the treatment of equations of only the second degree that such expressions appeared for the first time. Complex quantities are equally important in analysis, and we shall now state precisely what meaning is to be attached to the expression *a function of a complex variable.*

2. Continuous functions of a complex variable. A complex quantity $z = x + yi$, where x and y are two real and independent variables, is a complex variable. If we give to the word *function* its most general meaning, it would be natural to say that every other complex quantity u whose value depends upon that of z is a *function of z.*

Certain familiar definitions can be extended directly to these functions. Thus, we shall say that a function $u = f(z)$ is continuous if the absolute value of the difference $f(z+h) - f(z)$ approaches zero when the absolute value of h approaches zero, that is, if to every positive number ϵ we can assign another positive number η such that

$$|f(z+h) - f(z)| < \epsilon,$$

provided that $|h|$ be less than η.

A series,

$$u_0(z) + u_1(z) + \cdots + u_n(z) + \cdots,$$

whose terms are functions of the complex variable z is *uniformly convergent* in a region A of the plane if to every positive number ϵ we can assign a positive integer N such that

$$|R_n| = |u_{n+1}(z) + u_{n+2}(z) + \cdots| < \epsilon$$

for all the values of z in the region A, provided that $n \geqq N$. It can be shown as before (Vol. I, § 31, 2d ed.; § 173, 1st ed.) that if a series is uniformly convergent in a region A, and if each of its terms is a continuous function of z in that region, its sum is itself a continuous function of the variable z in the same region.

Again, a series is uniformly convergent if, for all the values of z considered, the absolute value of each term $|u_n|$ is less than the corresponding term v_n of a convergent series of real positive constants. The series is then both absolutely and uniformly convergent.

Every continuous function of the complex variable z is of the form $u = P(x, y) + Q(x, y)i$, where P and Q are real continuous functions of the two real variables x, y. If we were to impose no other restrictions, the study of functions of a complex variable would amount simply to a study of a pair of functions of two real variables, and the use of the symbol i would introduce only illusory simplifications. In order to make the theory of functions of a complex variable present some analogy with the theory of functions of a real variable, we shall adopt the methods of Cauchy to find the conditions which the functions P and Q must satisfy in order that the expression $P + Qi$ shall possess the fundamental properties of functions of a real variable to which the processes of the calculus apply.

3. Analytic functions. If $f(x)$ is a function of a real variable x which has a derivative, the quotient

$$\frac{f(x+h) - f(x)}{h}$$

approaches $f'(x)$ when h approaches zero. Let us determine in the same way under what conditions the quotient

$$\frac{\Delta u}{\Delta z} = \frac{\Delta P + i\Delta Q}{\Delta x + i\Delta y}$$

will approach a definite limit when the absolute value of Δz approaches zero, that is, when Δx and Δy approach zero independently. It is easy to see that this will not be the case if the functions $P(x, y)$ and $Q(x, y)$ are any functions whatever, for the limit of the quotient $\Delta u/\Delta z$ depends in general on the ratio $\Delta y/\Delta x$, that is, on the way in which the point representing the value of $z + h$ approaches the point representing the value of z.

Let us first suppose y constant, and let us give to x a value $x + \Delta x$ differing but slightly from x; then

$$\frac{\Delta u}{\Delta z} = \frac{P(x + \Delta x, y) - P(x, y)}{\Delta x} + i\frac{Q(x + \Delta x, y) - Q(x, y)}{\Delta x}.$$

In order that this quotient have a limit, it is necessary that the functions P and Q possess partial derivatives with respect to x, and in that case

$$\lim \frac{\Delta u}{\Delta z} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}.$$

Next suppose x constant, and let us give to y the value $y + \Delta y$; we have

$$\frac{\Delta u}{\Delta z} = \frac{P(x, y + \Delta y) - P(x, y)}{i\Delta y} + \frac{Q(x, y + \Delta y) - Q(x, y)}{\Delta y},$$

and in this case the quotient will have for its limit

$$\frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y}$$

if the functions P and Q possess partial derivatives with respect to y. In order that the limit of the quotient be the same in the two cases, it is necessary that

$$(1) \qquad \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

Suppose that the functions P and Q satisfy these conditions, and that the partial derivatives $\partial P/\partial x$, $\partial P/\partial y$, $\partial Q/\partial x$, $\partial Q/\partial y$ are continuous functions. If we give to x and y any increments whatever, Δx, Δy, we can write

$$\begin{aligned}\Delta P &= P(x + \Delta x, y + \Delta y) - P(x + \Delta x, y) + P(x + \Delta x, y) - P(x, y)\\ &= \Delta y P'_y(x + \Delta x, y + \theta\Delta y) + \Delta x P'_x(x + \theta'\Delta x, y)\\ &= \Delta x[P'_x(x, y) + \epsilon] + \Delta y[P'_y(x, y) + \epsilon_1],\end{aligned}$$

where θ and θ' are positive numbers less than unity; and in the same way

$$\Delta Q = \Delta x[Q'_x(x, y) + \epsilon'] + \Delta y[Q'_y(x, y) + \epsilon'_1],$$

where ϵ, ϵ', ϵ_1, ϵ'_1 approach zero with Δx and Δy. The difference $\Delta u = \Delta P + i\Delta Q$ can be written by means of the conditions (1) in the form,

$$\begin{aligned}\Delta u &= \Delta x\left(\frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}\right) + \Delta y\left(-\frac{\partial Q}{\partial x} + i\frac{\partial P}{\partial x}\right) + \eta\Delta x + \eta'\Delta y \\ &= (\Delta x + i\Delta y)\left(\frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}\right) + \eta\Delta x + \eta'\Delta y,\end{aligned}$$

where η and η' are infinitesimals. We have, then,

$$\frac{\Delta u}{\Delta z} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} + \frac{\eta\Delta x + \eta'\Delta y}{\Delta x + i\Delta y}.$$

If $|\eta|$ and $|\eta'|$ are smaller than a number α, the absolute value of the complementary term is less than 2α. This term will therefore approach zero when Δx and Δy approach zero, and we shall have

$$\lim \frac{\Delta u}{\Delta z} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x}.$$

The conditions (1) are then necessary and sufficient in order that the quotient $\Delta u/\Delta z$ have a unique limit for each value of z, provided that the partial derivatives of the functions P and Q be continuous. The function u is then said to be an *analytic* function * of the variable z, and if we represent it by $f(z)$, the derivative $f'(z)$ is equal to any one of the following equivalent expressions:

$$(2) \quad f'(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y} = \frac{\partial P}{\partial x} - i\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial y} + i\frac{\partial Q}{\partial x}.$$

It is important to notice that neither of the pair of functions $P(x, y)$, $Q(x, y)$ can be taken arbitrarily. In fact, if P and Q have derivatives of the second order, and if we differentiate the first of the relations (1) with respect to x, and the second with respect to y, we have, adding the two resulting equations,

$$\Delta P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0.$$

---

* Cauchy made frequent use of the term *monogène*, the equivalent of which, *monogenic*, is sometimes used in English. The term *synectique* is also sometimes used in French. We shall use by preference the term *analytic*, and it will be shown later that this definition agrees with the one which has already been given (I, § 197, 2d ed.; § 191, 1st ed.)

We can show in the same way that $\Delta Q = 0$. The two functions $P(x, y)$, $Q(x, y)$ must therefore be a pair of solutions of Laplace's equation.

Conversely, any solution of Laplace's equation may be taken for one of the functions P or Q. For example, let $P(x, y)$ be a solution of that equation; the two equations (1), where Q is regarded as an unknown function, are compatible, and the expression

$$u = P(x, y) + i\left[\int_{(x_0, y_0)}^{(x, y)}\left(\frac{\partial P}{\partial x}\,dy - \frac{\partial P}{\partial y}\,dx\right) + C\right],$$

which is determined except for an arbitrary constant C, is an analytic function whose real part is $P(x, y)$.

It follows that the study of analytic functions of a complex variable z amounts essentially to the study of a pair of functions $P(x, y)$, $Q(x, y)$ of two real variables x and y that satisfy the relations (1). It would be possible to develop the whole theory without making use of the symbol i.*

We shall continue, however, to employ the notation of Cauchy, but it should be noticed that there is no essential difference between the two methods. Every theorem established for an analytic function $f(z)$ can be expressed immediately as an equivalent theorem relating to the pair of functions P and Q, and conversely.

Examples. The function $u = x^2 - y^2 + 2xyi$ is an analytic function, for it satisfies the equations (1), and its derivative is $2x + 2yi = 2z$; in fact, the function is simply $(x + yi)^2 = z^2$. On the other hand, the expression $v = x - yi$ is not an analytic function, for we have

$$\frac{\Delta v}{\Delta z} = \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y} = \frac{1 - i\dfrac{\Delta y}{\Delta x}}{1 + i\dfrac{\Delta y}{\Delta x}},$$

and it is obvious that the limit of the quotient $\Delta v/\Delta z$ depends upon the limit of the quotient $\Delta y/\Delta x$.

If we put $x = \rho\cos\omega$, $y = \rho\sin\omega$, and apply the formulæ for the change of independent variables (I, § 63, 2d ed.; § 38, 1st ed., Ex. II), the relations (1) become

$$\frac{\partial P}{\partial \omega} = -\rho\frac{\partial Q}{\partial \rho}, \qquad \frac{\partial Q}{\partial \omega} = \rho\frac{\partial P}{\partial \rho}, \tag{3}$$

and the derivative takes the form

$$f'(z) = \left(\frac{\partial P}{\partial \rho} + i\frac{\partial Q}{\partial \rho}\right)(\cos\omega - i\sin\omega).$$

---

* This is the point of view taken by the German mathematicians who follow Riemann.