Memoirs of the American Mathematical Society

Number 175

Mark Alan Mostow

Continuous cohomology of spaces with two topologies

Published by the

AMERICAN MATHEMATICAL SOCIETY

Providence, Rhode Island

VOLUME 7 · NUMBER 175 (second of 4 numbers) · SEPTEMBER 1976

AMS(MOS) subject classifications (1970)

Primary: 55B30, 55B35, 55B40, 57D30

Secondary: 18F20, 18H10, 22-00, 22A99, 58A10

Key words and phrases

Continuous cohomology of spaces with two topologies, continuous cohomology, spaces with two topologies, smooth cohomology, Borel cohomology, cohomology theories for foliated manifolds, continuous cohomology of groups, category of pairs, homotopy of pairs, homotopy invariants of pairs, Reeb foliation, axioms for continuous cohomology, unwinding of simplicial modules, relative deRham complex.

Library of Congress Cataloging in Publication Data **CIP**

Mostow, Mark Alan, 1948-
Continuous cohomology of spaces with two topologies.

(Memoirs of the American Mathematical Society ; no. 175)
Thesis--Harvard University, 1975.
Bibliography: p.
1. Homology theory. 2. Topological spaces.
3. Foliations (Mathematics) I. Title. II. Series : American Mathematical Society. Memoirs ; no. 175.
QA3.A57 no. 175 [QA612.3] 510.8s [514'.2] 76-25187
ISBN 0-8218-2175-X

Table of contents

Abstract

This paper investigates the continuous cohomology of spaces with two topologies, as introduced by R. Bott and A. Haefliger. Given a space X with a second, finer topology X' , they defined the continuous cohomology $H_c^*(X' \to X)$ to be the cohomology of the complex of real-valued singular cochains on X' which are continuous with respect to the topology of X. The present paper studies other possible definitions of continuous cohomology and compares them by computing examples and by introducing four axioms which are shown to characterize the continuous cohomology of a foliated manifold (with its ordinary and leaf topologies). These axioms for a continuous cohomology theory T^* are

1. Homotopy invariance in the category of pairs
2. T^* of a topological sum = direct product of T^* of components
3. T^* satisfies a generalized Mayer-Vietoris Theorem expressing $T^*(X' \to X)$ as the limit of a spectral sequence with $E_1^{pq} = T^q(U_p' \to U_p)$, where U_p is the topological sum of the (p+1)-fold intersections of an open cover of X
4. (Normalization) $T^q(X_d \to X) = C(X)$ if q=0, and 0 if q>0, where X is any paracompact space, X_d is X with the discrete topology, and C(X) = {continuous real-valued functions on X}.

It is shown that a sheafified version T_c^* of H_c^* (envisioned but not published by Bott and Haefliger) satisfies all four axioms on a suitable category (the obstacle to proving Axiom 3 for H_c^* is discussed). The functor $T_{loc}^*(X' \to X)$ = (def.) (sheaf cohomology of the sheaf on X of continuous real-valued functions on X which are locally constant on X') is shown to satisfy the axioms also. $T_c = T_{loc}$ on foliated manifolds, but an example is given for which $T_c \neq T_{loc}$. Bott and Haefliger's assertion that $H_c^*(X \to X) \approx H_{singular}^*(X;\mathbb{R})$ for "reasonable" spaces X with one topology is proved, but with H_c^* replaced by T_c^* . Their assertion that $H_c^*(B(G_d) \to BG) = H_{cve}^*(G;\mathbb{R})$ is also proved for T_c^* in place of H_c^* ; here BG is the Milnor classifying space of a locally compact group G, $B(G_d)$ is the same construction for the discrete group G, and H_{cve}^* is continuous (Van Est) cohomology of a topological group. Smooth and Borel analogues of continuous cohomology are developed. A surprising result is that smooth cohomology need <u>not</u> map to continuous cohomology. The continuous, smooth, and Borel cohomologies of the torus foliated by lines of constant slope and of the Reeb foliation on S^3 are computed, as are the cohomologies of product foliations, fibrations, and foliations transverse to the fibers of flat bundles.

Acknowledgment

This paper is, with minor changes, my doctoral dissertation, which was submitted at Harvard University in May, 1975.

I owe a great debt to my thesis advisor, Professor Raoul Bott, for introducing me to this problem, for his valuable mathematical suggestions and for his insistence that I include examples and discussion to make the exposition more readable.

Introduction

The object of this thesis is to study the various possible definitions and the properties of continuous cohomology of spaces with two topologies, as introduced by R. Bott and A. Haefliger [2].

The term continuous cohomology has had several meanings in mathematics. One can define the continuous cohomology $H_c^*(X)$ of a space X to be the cohomology of the complex of continuous singular cochains on X with values in $\mathbb{R}$, where a cochain h is called continuous if $h(\sigma)$ varies continuously as the singular simplex σ is moved continuously in X. A related theory using continuous Alexander-Spanier cochains was defined by S. T. Hu [22]. Not surprisingly, however, $H_c^*(X) = H^*(X;\mathbb{R})$ for reasonable spaces X (if we work with a sheafified variant of H_c^*, at least), so that continuous cohomology is not a very interesting concept for ordinary spaces.

The continuous cohomology of a topological or Lie group G was defined and studied by S. T. Hu [22, p. 13], W. T. Van Est [46], G. D. Mostow [36] and G. Hochschild [21]; it differs from the Eilenberg-MacLane cohomology of the group G only in using continuous (or smooth) rather than arbitrary functions on the inhomogeneous complex

$$NG = pt. \leftleftarrows G \leftleftarrows G \times G \ \ldots \quad .$$

That is,

$$H_c^*(G;\mathbb{R}) = (\text{def.})\ H(\text{Map}(NG, \mathbb{R})) \quad .$$

More recently, R. Bott and A. Haefliger [2] have observed that if a space X' has a second, coarser topology X (fewer open sets), then one can define the continuous cohomology $H_c^*(X' \to X)$ of the space with two topologies, which we denote $(X' \to X)$, to be the cohomology of the subcomplex of $S^*(X'; \mathbb{R})$ consisting of cochains which are continuous relative to the topology of X. (An alternate homotopy theoretic definition was subsequently proposed by G. Segal (§2 and [40]).) The relation Bott and Haefliger observed between their concept and the continuous cohomology of groups is that if BG is the (simplicially constructed) classifying space of G and G_d is G with the discrete topology, then

$$H_c^*(B(G_d) \longrightarrow BG) = H_c^*(G; \mathbb{R}) \qquad .$$

They proposed that one could relate the continuous cohomology of Haefliger's category Γ of foliation theory, with its sheaf and jet topologies Γ and J, to the continuous cohomology of the space $(B\Gamma \to BJ)$ with two topologies.

My point of departure was to study and compare different possible definitions of continuous cohomology of spaces with two topologies. Bott and Haefliger realized that their theory should be sheafified, and this gives rise to a theory T_c (§ 3). Another theory is $T_{loc}(X' \to X)$, which is defined to be the cohomology of the sheaf (on X) of germs of continuous real valued functions on X which are locally constant on X'. In order to facilitate the comparison of these two theories, I showed that both T_c and T_{loc} satisfy four quite reasonable axioms for continuous cohomology (§§2,3), and that all theories satisfying these axioms must agree on $(BG_d \to BG)$ (§7).

Another large class of examples of spaces with two topologies is provided by foliated manifolds. Namely, if M is a manifold with a foliation F, let

M^F denote the set M topologized as the disjoint union of the leaves, each topologized as an abstract manifold ; $(M^F \to M)$ is then a space with two topologies. In §6 we show that all theories satisfying the axioms agree on foliated manifolds $(M^F \to M)$.

We see, then, that from the point of view of the familiar examples $(BG_d \to BG)$ and $(M^F \to M)$, there is no way to distinguish between different continuous cohomology theories (which satisfy the axioms). However, if we intend to define the continuous cohomology of such spaces as $(B\Gamma \to BJ)$, which are quite pathological, then it is essential to know if T_c and T_{loc} ever disagree, so that we may decide which is the "right" theory to use on $(B\Gamma \to BJ)$. In §8 we present two examples, neither very pathological, which distinguish different cohomology theories. This shows that one must really choose a theory to extend the study of continuous cohomology to new situations. We do not make such a choice here, but we do discuss the kinds of conditions which make theories agree or disagree.

In §9 we compute the continuous cohomology of several foliated manifolds, including the torus foliated by lines of rational or irrational slope, and the Reeb foliation on S^3. The answers show that continuous cohomology is an interesting invariant of a foliation which is quite sensitive to the manner in which the leaves twist in the manifold. We point out, however, that this invariant is quite large ; it is typically a subgroup or quotient of the group of continuous real-valued functions on certain transversals of the foliation. Also, although it is invariant under homeomorphism and under homotopy of spaces with two topologies, it is not invariant under the usual relations of homotopy or concordance of foliations (these homotopy relations are compared in §1).

Sections 4 and 5 discuss smooth and Borel analogues of continuous cohomology. We see by the examples in sections 8 and 9 that these theories are related to, but qualitatively different from, continuous cohomology. In particular, the Borel cohomology of a foliated manifold measures the ergodicity of the foliation, in some sense, while the Borel cohomology of $(BG_d \to BG)$ equals the Borel cohomology of G defined by Calvin Moore [34]. The smooth cohomology of a foliated manifold is the cohomology of differential forms on the leaves which vary smoothly on the manifold; this coincides with the Lie algebra cohomology of the vector fields on the leaves with values in the smooth functions on the manifold, as defined by Kamber and Tondeur [24]. In the special case of a foliation defined by a fibration, this equals the cohomology of the relative deRham complex of algebraic geometry.

This work makes extensive use of sheaf theory; consequently, I have outlined some basic concepts of that theory in §3 . We also use some of the theory of simplicial objects, and in particular the concept of unwinding a simplicial object. The unwinding of a category was defined by G. Segal in [38], and the unwound (or Milnor) realization of an arbitrary simplicial space by T. tom Dieck [45], but I do not know any reference for the unwinding of a simplicial or cosimplicial module, which I have defined and studied in Appendix A. Appendix B is a compendium of useful criteria for paracompactness; this is an essential question for us because the theory $T_c(X' \to X)$ satisfies the four axioms of continuous cohomology only when X is paracompact. In particular, we prove that BJ (and J) are paracompact, so that T_c satisfies the axioms of continuous cohomology on $(\Gamma \to J)$ and $(B\Gamma \to BJ)$ (T_{loc} also satisfies the axioms on these spaces).

Section 1 establishes our definitions and notation for discussing spaces with two topologies.

I would like to express my appreciation to Professor Raoul Bott, my thesis advisor, who suggested this problem to me and made invaluable suggestions, especially the idea of sheafifying his and Haefliger's continuous cochain theory. I thank him and Professors Joel Wolf and Robert Jackson for serving on my thesis committee and presiding at my thesis defense. I also wish to thank Judy Moore for her excellent job of typing and Rebecca M.K. Nemser for her illustrations. Finally, I thank Professor James Stasheff and the Memoirs referee for their suggestions for improving the manuscript.

CONTINUOUS COHOMOLOGY OF SPACES WITH TWO TOPOLOGIES

§1 Spaces with two topologies

In many problems in mathematics one encounters a point set X endowed with two topologies, a fine topology (more open sets) and a coarse topology (fewer open sets). Well-known examples include the strong and weak topologies on a Hilbert space, the subspace and Zariski topologies on a projective variety X embedded in complex projective n-space, and the Whitehead (weak) and metric topologies on a simplicial complex [23 , p. 121].

In this thesis, I shall be more interested in examples that arise in the theory of foliations. These are of two principal types :

1. Let M be a manifold with a foliation F. Let M^F denote the set M topologized as the disjoint union of the leaves of F, each topologized as an abstract manifold (<u>not</u> as a subspace of M). Here M is the space with the coarse topology and M^F has the fine topology.

2. For any space X let X_d denote X with the discrete topology. Let G be a topological group and let BG be its Milnor classifying space (See Appendix). Then $B(G_d)$ is the same set as BG, but with a finer topology. (The relationship of this example to foliations is that while BG classifies G-bundles, $B(G_d)$ classifies G-bundles with locally constant transition functions, and these have a canonical foliation transversal to the fibers.)

Received by the editors December 4, 1975, and in revised form February 21, 1976.

An announcement of this paper appeared in Notices of the AMS, August, 1975, page A-529, abstract number 75T-G79.

A variant of the second example is given by Γ_q , Haefliger's topological category of germs of local diffeomorphisms of $\mathbb{R}^q$ (which we will always assume C^∞) ([16, p. 146]). The standard, sheaf, or fine topology on Γ_q is as a subspace of the sheaf of $\mathbb{R}^q$-valued functions on $\mathbb{R}^q$. A second, coarse, topology on Γ_q is the jet topology, obtained by mapping Γ_q to J_q, the space of ∞-jets of local diffeomorphisms of $\mathbb{R}^q$, and pulling back (see below) the Whitney C^∞ topology on J_q (two jets are close if their source, target, and a finite number of partial derivatives are close). As in example 2, we can take the Milnor classifying space (of a category; see Appendix), and we obtain $B\Gamma_q$, a space with two topologies.

In all our examples, we defined two topologies on a space X' by means of a continuous map $i : X' \longrightarrow X$, which was usually bijective as a set map (e. g. $M^F \longrightarrow M$ and $B(G_d) \longrightarrow BG$), but sometimes was not, as in the case $i : \Gamma_q \longrightarrow J_q$. We now formalize this procedure.

Definition. Let $i : X' \longrightarrow X$ be a map of sets, and suppose that X is endowed with a topology. Then the pullback topology on X' is $\{f^{-1}U | U \text{ open in } X\}$.

The pullback topology makes i continuous and satisfies the universal property that for all spaces W and all set maps $g : W \longrightarrow X'$, g is continuous if and only if ig is continuous.

We now define a <u>space with two topologies</u> to be a continuous map of spaces $i : X' \longrightarrow X$, which we denote by one of the three notations

$$X! = (i : X' \to X) = (X' \to X) \quad .$$

We shall not require, as G. Segal does [40], that i be bijective (as a map of sets). The given topology on X' will be called the <u>fine topology</u>, and the pullback topology on X' will be called the <u>coarse topology</u>. (If i is bijective, then X' with the pullback topology is just the space X.)

<u>Examples of notation</u>. 1. $(i : \Gamma \to J)$ and $(B\Gamma \to BJ)$.

2. $M! = (M^F \to M)$ is a <u>foliated manifold</u>.

<u>Definition</u>. For any space X, X_D will denote the space with two topologies $(X_d \to X)$ (recall that $X_d = X$ with the discrete topology).

<u>Examples</u>. 1. $G_D = (G_d \to G)$, G a topological group.

2. $B(G_D) = (B(G_d) \longrightarrow BG)$.

We now develop a vocabulary for discussing spaces with two topologies. We follow the terminology of P. Hilton and B. Eckmann [20, p. 11], who introduced the category of pairs, and of G. Segal [40].

<u>Definition</u>. Let C_2 be the category whose objects are continuous maps $(i : X' \to X)$ of topological spaces (also written $X!$ or $(X' \to X)$) and whose morphisms $f! : X! \longrightarrow Y!$ are commutative diagrams of continuous maps

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \Big\downarrow & & \Big\downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

also written

$$(f', f) : (X' \to X) \longrightarrow (Y' \to Y) \quad .$$

C_2 will be called the <u>category of spaces with two topologies</u> or the <u>category of pairs</u>.

Observe that a morphism $f! : X! \longrightarrow Y!$ in C_2 is continuous as a map from X' to Y' if these spaces are either both given the fine (given) topology or both given the coarse (pullback) topology; the latter statement follows from the universal property of the pullback topology.

<u>Example of a morphism in C_2</u>. Let M, N be manifolds with foliations F, G resp. Then a morphism

$$(f', f) : (M^F \to M) \longrightarrow (N^G \to N)$$

in C_2 is a continuous map $f : M \longrightarrow N$ which maps each leaf of F continuously into a leaf of G.

In C_2, we can define the analogues of such topological notions as subspaces, products, and homotopies.

<u>Definition</u>. If $U!$ and $X! = (i : X' \to X)$ are objects in C_2 then we call $U!$ a <u>subspace</u> of $X!$ and write $U! \subset X!$ if $U \subset X$ and

$U' = i^{-1}U$ topologized as a subspace of X' ; it is an <u>open subspace</u> if U is open in X (and hence U' is open in X') . An <u>open cover(ing)</u> of $X!$ is a collection $\mathfrak{U}! = \{U_a!\}_{a\in A}$ of open subspaces such that $\cup_a U_a = X$.

<u>Disjoint unions</u> (i.e. <u>topological sums</u>) have the obvious meaning : $\coprod_a X_a! = (\coprod X'_a \to \coprod X_a)$.

We denote by $X! \times Y!$ the object $(X' \times Y' \to X \times Y)$, and by $X! \times Y$ the object $(X' \times Y \to X \times Y)$ in C_2 . In particular, letting I denote the closed unit interval, a <u>homotopy</u> of two morphisms

$$f_0! \text{ and } f_1! : X! \longrightarrow Y!$$

in C_2 is a morphism

$$\{f_t!\} = f! : X! \times I \longrightarrow Y! \quad .$$

In other words, a homotopy in C_2 is a pair of homotopies $f' : X' \times I \longrightarrow Y'$ and $f : X \times I \longrightarrow Y$ which are compatible with the maps $X' \longrightarrow X$ and $Y' \longrightarrow Y$. We write $f_0! \sim f_1!$ if $f_0!$ and $f_1!$ are homotopic in C_2 .

<u>Homotopy equivalences</u>, <u>deformation retractions</u> (also called <u>deformations</u>) and <u>homeomorphisms</u> in C_2 are defined in terms of homotopies and morphisms in C_2 by analogy with the ordinary case. (The word <u>ordinary</u> will always refer to spaces with one topology.) For example, a morphism $f! : X! \longrightarrow Y!$ is a homotopy equivalence in C_2 if and only if there exists a morphism $g! : Y! \longrightarrow X!$ satisfying $f! \circ g! \sim id_{Y!}$, $g! \circ f! \sim id_{X!}$ (see also Example 2 and Remarks 1 and 2).

<u>Examples</u>

1. If $(M^F \to M) \in C_2$ is a foliated manifold, then an open subspace $U!$ of $(M^F \to M)$ is an open subspace $U \subset M$, regarded as a space with two topologies by giving it the foliation $F|U$.

2. The foliated manifolds $\mathbb{R}_D$ and $\mathbb{R}_D \times \mathbb{R}$ are homotopy equivalent objects in C_2. More generally, if $h : A \times I \longrightarrow B$ is an ordinary homotopy, and $X! \in C_2$, then

$$\mathrm{id} \times h_0 \text{ and } \mathrm{id} \times h_1 : X! \times A \longrightarrow X! \times B$$

are homotopic morphisms in C_2.

<u>Remarks</u>

1. The definition of homotopy is quite restrictive. For example, if $(f', f) : (X' \to X) \longrightarrow (Y' \to Y)$ is a morphism in C_2, and if f' and f are ordinary homotopy equivalences of (ordinary) spaces, then (f', f) is <u>not</u> a homotopy equivalence in C_2 unless there exist homotopy inverses g' of f' and g of f, and corresponding homotopies, such that

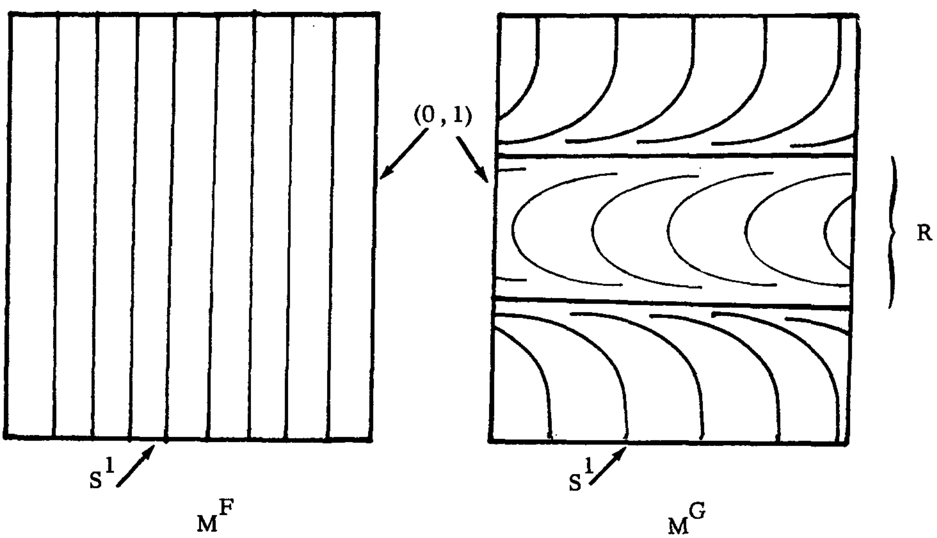
$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \longrightarrow & X \end{array}$$

commutes, and such that the two similar diagrams formed from the homotopies of $f'g'$, fg, $g'f'$, and gf to their respective identity maps also commute.

2. The concept of homotopy in C_2 is not to be confused with homotopy

or concordance of foliations [6]. In C_2 we do not speak of two objects (for example, two foliated manifolds) being homotopic. However, we can ask if two foliations on the same manifold which are homotopic or concordant as foliations are necessarily homotopy equivalent in C_2. The answer is no, as the following example shows.

Let M be a cylinder, $M = S^1 \times (0,1)$. Let F be the foliation of M whose leaves are pt. $\times (0,1)$. One can modify F by introducing a Reeb component R and sweeping the leaves of F around the two boundary leaves of R, which are circles; let G be the resulting foliation of M (see Figure 1).



(In both pictures, identify right and left edges to obtain the cylinder.)

Figure 1

Now F and G are both homotopic and concordant to each other as foliations [6]. Suppose that $(M^F \to M)$ and $(M^G \to M)$ were homotopy equivalent in C_2. Then M^F and M^G would be homotopy equivalent in the ordinary sense. But M^F is a disjoint union of intervals, while M^G is a disjoint union of two circles plus some other leaves, so this is impossible.

3. It will sometimes be inconvenient to define homotopies as morphisms from $X! \times I$ to $Y!$. For example, if $X!$ is a foliated manifold, then $X! \times I$ is not a foliated manifold (just because it has a boundary). In this case we will sometimes replace I by $\mathbb{R}$ in the definition of homotopy ; it is clear that two morphisms are homotopic in one definition if and only if they are in the other.

4. Identifying the space X with the object $(id : X \to X)$ embeds Top (= topological spaces) as a full subcategory of C_2 on which the terms subspace, cover, and homotopy (as defined in C_2) assume their ordinary meanings.

§ 2 Axioms for a continuous cohomology theory

Just as we use singular and other cohomology theories to study homotopy invariant properties of ordinary spaces, we can use cohomology theories on C_2 to study properties of spaces with two topologies which are invariant under homotopies in C_2. In this section we look at several such theories, and then introduce axioms for continuous cohomology theories on C_2.

One cohomology theory on C_2 was defined by P. Hilton and B. Eckmann [20, p. 25] by homotopy theoretic methods. It is essentially a generalization of the relative singular cohomology of a pair; for example, if $X' \subset X$, then $H^*(X' \to X) = H^*(X, X')$ in their theory. In particular, $H^*(\mathrm{id} : X \to X) = 0$.

A very different type of theory was introduced by R. Bott and A. Haefliger [2]. Given a continuous map $i : X' \longrightarrow X$, which in their examples is usually a bijection of sets, they define $S^*_c(X'/X)$ to be the subcomplex of $S^*(X'; \mathbb{R})$ consisting of (singular) cochains which are continuous relative to the topology of X (see also § 3). They then define the continuous cohomology of $(X' \to X)$ to be

$$H^*_c(X'/X) = HS^*_c(X'/X) \quad .$$

Letting $H^*_c(X)$ denote $H^*_c(X/X)$, they observe

1. There are natural maps

$$H^*_c(X) \longrightarrow H^*_c(X'/X) \longrightarrow H^*_c(X') \quad .$$

2. For reasonable spaces, the natural map $H^*_c(X) \to H^*(X; \mathbb{R})$ is an

isomorphism. [The second assertion was not proved in [2], and I do not know how to prove it; the difficulty in the "obvious" proof is discussed below (p.27, Remark 2). The assertion is true, however, if H_c^* is replaced by a sheafified theory T_c^*, as envisioned by Bott and Haefliger; see Theorems 2.6 and 3.6 below.]

Another approach to continuous cohomology was taken by G. Segal [40]. Given a topological abelian group A, he constructs an Eilenberg-Mac Lane space $K' = K(A_d, n) = B^n(A_d)$ and gives it a coarser topology $K = B^n(A)$ (using the simplical construction of BG as $|NG|$ (see Appendix)); then $K! = (K' \to K)$ is an object in $\mathcal{C}_2$. Given a space $X!$ with two topologies he defines

$$H^n(X!;A) = [X!,K!] = \{\text{homotopy classes (in } \mathcal{C}_2\text{) of morphisms: } X! \to K!\}.$$

For the special case of a foliated manifold $(M^F \longrightarrow M)$, a smooth cohomology theory has been defined by F. Kamber and P. Tondeur [24] as the Lie algebra cohomology of the sheaf of vector fields on the leaves with values in the sheaf of smooth functions on the manifold, which, as they note [24, Th. 4.27], equals the cohomology of the complex of differential forms on the leaves which vary smoothly on the manifold; in the case of a foliation defined by a fibration, this is the relative deRham complex of algebraic geometry.

In order to compare these various cohomology theories and to compute examples, I have found it convenient to make axioms for a continuous cohomology theory on C_2. My procedure for doing this was to abstract the properties of