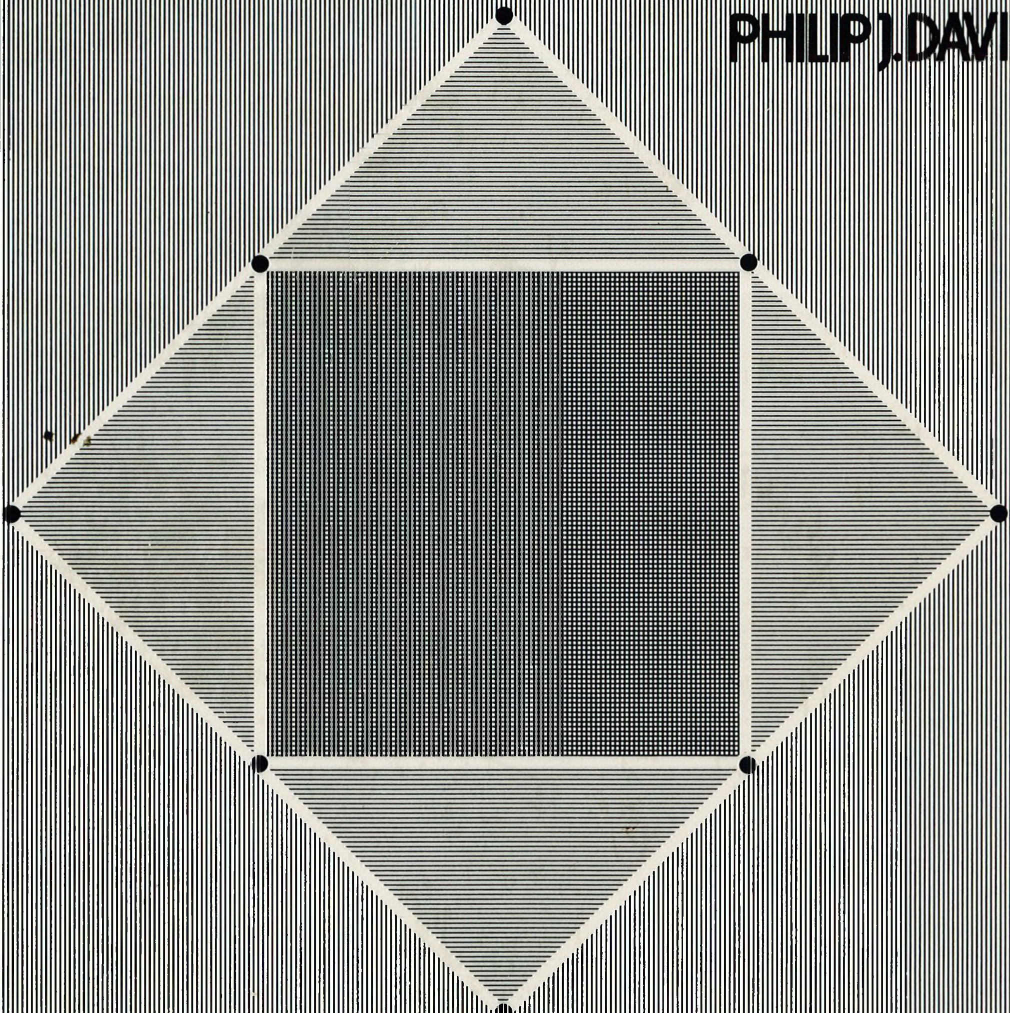
CIRCULANT MATRICES

PHILIP J. DAVIS



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Circulant matrices—those in which a basic row of numbers is repeated again and again, but with a shift in position—constitute a nontrivial but simple set of objects that can be used to practice, and ultimately to deepen, a knowledge of matrix theory. Circulant matrices have many connections to problems in physics, image processing, probability and statistics, numerical analysis, number theory, and geometry. Their built-in periodicity means that circulants tie in with Fourier analysis and group theory. Circulant theory is also relatively easy—practically every matrix-theoretic question for circulants can be resolved in "closed form."

This book is intended to serve as a general reference on circulants as well as to provide alternate or supplemental material for intermediate courses in matrix theory. It begins at the level of elementary linear algebra and increases in complexity at a gradual pace. First, a problem in elementary geometry is given to motivate the subsequent study. The complete theory is contained in Chapter 3, with further geometric applications presented in Chapter 4. Chapter 5 develops some of the generalizations of circulants. The final chapter places and studies circulants within the context of centralizers—taking readers to the fringes of current research in matrix theory.

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What is circular is eternal;
what is eternal is circular.

PREFACE

"Mathematics," wrote Alfred North Whitehead, "is the most powerful technique for the understanding of pattern and for the analysis of the relations of patterns." In its pursuit of pattern, however, mathematics itself exhibits pattern; the mathematics on the printed page often has visual appeal. Spatial arrangements embodied in formulae can be a source of mathematical inspiration and aesthetic delight.

The theory of matrices exhibits much that is visually attractive. Thus, diagonal matrices, symmetric matrices, (0, 1) matrices, and the like are attractive independently of their applications. In the same category are the circulants. A circulant matrix is one in which a basic row of numbers is repeated again and again, but with a shift in position. Circulant matrices have many connections to problems in physics, to image processing, to probability and statistics, to numerical analysis, to number theory, to geometry. The built-in periodicity means that circulants tie in with Fourier analysis and group theory.

A different reason may be advanced for the study of circulants. The theory of circulants is a relatively easy one. Practically every matrix-theoretic question for circulants may be resolved in "closed form." Thus the circulants constitute a nontrivial but simple set of objects that the reader may use to practice, and ultimately deepen, a knowledge of matrix theory.

Writers on matrix theory appear to have given circulants short shrift, so that the basic facts are rediscovered over and over again. This book is intended to serve as a general reference on circulants as well as to provide alternate or supplemental material for intermediate courses in matrix theory. The reader will need to be familiar with the geometry of the complex plane and with the elementary portions of matrix theory up through unitary matrices and the diagonalization of Hermitian matrices. In a few places the Jordan form is used.

This work contains some general discussion of matrices (block matrices, Kronecker products, the UDV theorem, generalized inverses). These topics have been included because of their application to circulants and because they are not always available in general books on linear algebra and matrix theory. More than 200 problems of varying difficulty have been included.

It would have been possible to develop the theory of circulants and their generalizations from the point of view of finite abelian groups and group matrices. However, my interest in the subject has a strong numerical and geometric base, which pointed me in the direction taken. The interested reader will find references to these algebraic matters.

Closely related to circulants are the Toeplitz matrices. This theory and its applications constitute a world of its own, and a few references will have to suffice. The bibliography also contains references to applications of circulants in physics and to the solution of differential equations.

I acknowledge the help and advice received from Professor Emilie V. Haynsworth. At every turn she has provided me with information, elegant proofs, and encouragement.

I have profited from numerous discussions with Professors J. H. Ahlberg and Igor Najfeld and should like to thank them for their interest in this essay. Philip R. Thrift suggested some important changes.

Thanks are also due to Gary Rosen for the Calcomp plots of the iterated n-gons and to Eleanor Addison for the figures. Katrina Avery, Frances Beagan, Ezoura Fonseca, and Frances Gajdowski have helped me enormously in the preparation of the manuscript, and I wish to thank them for this work, as well as for other help rendered in the past.

The Canadian Journal of Mathematics has allowed me to reprint portions of an article of mine and I would like to acknowledge this courtesy.

Finally, I would like to thank Beatrice Shube for inviting me to join her distinguished roster of scientific authors and the staff of John Wiley and Sons for their efficient and skillful handling of the manuscript.

Philip J. Davis

Providence, Rhode Island
April, 1979

CONTENTS

NOTATION

C	the complex number field
$C_{m\times n}$	the set of $m \times n$ matrices whose elements are in C
A^T	transpose of A
$\overline{A}$	conjugate of A
A^*	conjugate transpose of A
$A \otimes B$	direct (Kronecker) product of A and B
$A \circ B$	Hadamard (element by element) product of A and B
$A^{\div}$	Moore-Penrose generalized inverse of A
$r(A)$	rank of A

If A is square,

$\det(A)$	determinant of A
$\mathrm{tr}(A)$	trace of A
$\lambda(A)$	eigenvalues of A; individually or as a set
A^{-1}	inverse of A
$\rho(A)$	spectral radius of A

$$\mathrm{diag}(d_1, d_2, \ldots, d_n) = \mathrm{diag}(d_1, d_2, \ldots, d_n)^T = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}.$$

Z(A) centralizer of A (Section 6.2).

If A and B are square,

$A \oplus B = \mathrm{diag}(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ = direct sum of A and B

dg A $= \mathrm{dg}(a_{ij}) = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$

offdg A = A - dg A

Special Square Matrices

Subscripts are often (but not exclusively) used to designate the order of square matrices.

0 = zero = circ(0, 0, ..., 0)

I = identity = circ(1, 0, ..., 0)

π = fundamental permutation matrix = circ(0, 1, 0, ..., 0)

Q_r = r-circ(1, 0, ..., 0); $Q_k = Q_k(\lambda)$ = Jordan block

J = circ(1, 1, ..., 1); all entries of J are 1

$\Omega = \mathrm{diag}(1, w, w^2, \ldots, w^{n-1})$, $w = \exp(2\pi i/n)$, π=3.14...

$\eta = \pi\ \mathrm{diag}(-I_1, I_{n-1})$

Λ_k = diag(0, 0, ..., 0, 1, 0, ..., 0), 1 is in the kth position

F = Fourier matrix

$B_k = F^* \Lambda_k F$

Γ = (-1)-circ(1, 0, ..., 0)

K = counteridentity = (-1)-circ(0, 0, ..., 0, 1)

V = $V(z_0, z_1, \ldots, z_{n-1})$ = Vandermonde matrix

S = selector matrix

If $\phi(x) = x^n - a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \ldots - a_1 x - a_0$, the companion matrix of ϕ is

$$C_\phi = \begin{pmatrix} 0 & 1 & 0 & \ldots. & 0 \\ 0 & 0 & 1 & \ldots. & 0 \\ 0 & 0 & 0 & 1 \,.. & 0 \\ 0 & 0 & 0 & 0 \,.. & 1 \\ a_0 & a_1 & a_2 & \ldots. & a_{n-1} \end{pmatrix}.$$

Other notation

$$\begin{cases} \lambda^{\div} = \lambda^{-1} \text{ if } \lambda \neq 0 \\ \lambda^{\div} \equiv 0 \text{ if } \lambda = 0 \end{cases}$$

$\mathscr{P}$ = set of polynomials with scalar coefficients

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